MATH 245 F24, Exam 1 Solutions

1. Carefully define the following terms: factorial, floor.
The factorial, denoted !, is a function from $\mathbb{N}_0$ to $\mathbb{N}$ defined recursively as: $0! = 1$, and $n! = n \cdot (n-1)!$ (for $n \geq 1$). Let $x \in \mathbb{R}$. Then integer n is the floor of x if it satisfies $n \leq x < n+1$.

2. Carefully state the following theorems: Commutativity (for Propositions), Addition Semantic Theorem.
The Commutativity Theorem says: For any propositions p, q, we have $p \vee q \equiv q \vee p$ and also $p \wedge q \equiv q \wedge p$. The Addition Semantic Theorem says: For any propositions p, q, we have $p \vdash p \vee q$.

3. Prove Theorem 1.6, which says: If $n \in \mathbb{Z}$, then n is at least one of {odd, even}.
Direct proof, assume $n \in \mathbb{Z}$. Apply the Division Algorithm Theorem to $n, 2$, which gives integers q, r satisfying both $n = 2q + r$ and $0 \leq r < 2$. Now there are two cases.
Case $r = 0$: Now $n = 2q + 0 = 2q$, and $q \in \mathbb{Z}$, so n is even.
Case $r = 1$: Now $n = 2q + 1$, and $q \in \mathbb{Z}$, so n is odd.
In all cases, n is at least one of {odd, even}.

4. Prove or disprove: For all $a, b, c \in \mathbb{Z}$, if $ac|bc$ then $a|b$.
The statement is false, and requires a counterexample. Take $a = 2, b = 3, c = 0, k = 1$. We have $ack = 0 = bc$, so $ac|bc$ since $k \in \mathbb{Z}$. We will now prove that $a \nmid b$ by contradiction: if $a|b$ then there would be some $n \in \mathbb{Z}$ with $an = b$, then $2n = 3$, so $n = 1.5$. Since this is not an integer, we have a contradiction.

5. Simplify $\neg((p \to q) \to (r \wedge s))$ as much as possible, where only basic propositions are negated. Be sure to justify each step.
Step 1: By Thm 2.16 (Negated Conditional Interpretation), the expression becomes $(p \to q) \wedge \neg(r \wedge s)$.
Step 2: By De Morgan's Law, the expression becomes $(p \to q) \wedge ((\neg r) \vee (\neg s))$.

NOTE: Some of you used Conditional Interpretation to change $p \to q$ to $q \vee (\neg p)$ at the end. This is a matter of taste (is this simpler?). I graded either version as correct.

ALTERNATE SOLUTION:
Step 1: By Conditional Interpretation, the expression becomes $\neg((r \wedge s) \vee \neg(p \to q))$.
Step 2: By De Morgan's Law, the expression becomes $(\neg(r \wedge s)) \wedge \neg\neg(p \to q)$.
Step 3: By Double Negation, the expression becomes $(\neg(r \wedge s)) \wedge (p \to q)$.
Step 4: By De Morgan's Law, the expression becomes $((\neg r) \vee (\neg s)) \wedge (p \to q)$.

6. Let p, q, r be propositions. Without using truth tables, prove that: $p \to q, r \to q, p \vee r \vdash q$.
We must begin by assuming that $p \to q, r \to q, p \vee r$ are all true. Since $p \vee r$ is true, there are two cases:
Case p is true: Apply modus ponens with $p \to q$, concluding q is true.
Case r is true: Apply modus ponens with $r \to q$, concluding q is true.
In all cases, q is true.

7. Let $x \in \mathbb{R}$. Prove that if x^2 is irrational, then x is irrational.

Contrapositive proof. Assume that x is not irrational, i.e. x is rational. Then there are integers a, b, with $b \neq 0$, such that $x = \frac{a}{b}$. We now square to get $x^2 = \frac{a^2}{b^2}$. Both a^2 and b^2 are integers, and b^2 is not zero, so x^2 is rational, i.e. not irrational.

8. Suppose that p is prime. Prove that p^2 is composite.

Since p is prime, we know that $p \in \mathbb{Z}$ and also $p \geq 2$. Now, $p|p^2$ since $p \cdot p = p^2$ and p is an integer. Also, $p > 1$ since $p \geq 2 > 1$. Next, we multiply both sides of $p > 1$ by the positive p to get $p^2 > p$. Combining, we get $1 < p < p^2$. Lastly p^2 is an integer (since p is an integer) and $p^2 > p \geq 2$ so $p^2 \geq 2$. Hence p is composite.

9. Prove that, for all propositions p, q, we have $(p \uparrow q) \wedge (p \rightarrow q) \equiv \neg p$.

Let p, q be arbitrary propositions. In the truth table at right, the fifth and sixth columns agree, so $(p \uparrow q) \wedge (p \rightarrow q) \equiv \neg p$.

p	q	$p \uparrow q$	$p \rightarrow q$	$(p \uparrow q) \wedge (p \rightarrow q)$	$\neg p$
T	T	F	T	F	F
T	F	T	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

10. Prove or disprove: $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ (x+1)^2 < y < (x+2)^2$.

The statement is false. We need to prove $\neg \forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ (x+1)^2 < y < (x+2)^2$, i.e. $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ \neg\big((x+1)^2 < y < (x+2)^2\big)$. Hence, we need to prove $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ (x+1)^2 \geq y \vee y \geq (x+2)^2$.

SOLUTION 1: Choose $x = -1$, and let $y \in \mathbb{Z}$ be arbitrary. We have two cases, either y is positive or it isn't.
Case y is positive: $y > 0$, but $y \in \mathbb{Z}$, so $y \geq 1$ (by Thm 1.12). i.e. $y \geq (x+2)^2$, so by addition $(x+1)^2 \geq y \vee y \geq (x+2)^2$.
Case y isn't positive: $y \leq 0$, so $y \leq (x+1)^2$, so by addition $(x+1)^2 \geq y \vee y \geq (x+2)^2$.
In all cases, $(x+1)^2 \geq y \vee y \geq (x+2)^2$.

SOLUTION 2: Choose $x = -3$, and let $y \in \mathbb{Z}$ be arbitrary. We will prove that $(x+1)^2 < y < (x+2)^2$ is impossible (using the contradiction semantic theorem); if this were true then $(-2)^2 < y < (-1)^2$, and hence $4 < y < 1$, and hence $4 < 1$. But $4 \not< 1$. Hence $\neg((x+1)^2 < y < (x+2)^2)$, as desired.